ISRAEL JOURNAL OF MATHEMATICS, Vol. 29, Nos. 2-3, 1978

ON NORMAL p-SUBGROUPS WITH LARGE CENTERS WHICH CANNOT BE CONTAINED IN THE FRATTINI SUBGROUP

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ABSTRACT

Let K be a characteristic subgroup of a p-group H such that H induces on K a sufficiently large group of automorphisms. Then H cannot be embedded as a normal subgroup contained in the Frattini subgroup in any finite group. The group H may have a large center without any characteristic subgroup of H properly contained in it. Examples are given for such H with $Z(H)$ elementary abelian of arbitrary dimension.

1. Introduction

All groups in this note are assumed to be finite. By $\mathfrak{X}$ we denote the class of nilpotent groups which cannot be embedded as normal subgroups contained in the Frattini subgroup in some finite group. Gaschütz [1] gave a sufficient condition for a nilpotent group to lie in $\mathfrak{X}$, which was used by various authors. There was one result of Stitzinger [10], Hill and Wright [3] which was basic for recent work on the problem of finding further sufficient conditions for a nilpotent group to lie in $\mathfrak{X}$.

THEOREM (Stitzinger [10], Hill and Wright [3]). *Let H be a p-group with an abelian characteristic subgroup A such that $|A \cap Z(H)| = p$. $A \leqq Z_2(H)$ but $A \not\leqq Z(H)$. Then $H \in \mathfrak{X}$.*

Recently Hill [2] proved that for an odd prime p a p-group H with a characteristic subgroup K such that $|K| > |K \cap Z(H)| = p$ is in $\mathfrak{X}$, and Makan [6] classified all p-groups, p odd or even, with cyclic center that lie in $\mathfrak{X}$.

All these theorems require that the p-group H has a characteristic subgroup of order p that lies in $\mathfrak{X}$. To make further progress it seems to be necessary to

Received June 14, 1976 and in revised form January 25, 1977

have conditions which are free from this restriction. It is therefore the aim of this article to give such results.

2. Results

Following the concept of Stitzinger, Hill and Wright we consider a characteristic subgroup K of H. In addition we consider $\bar{H}$, the group of automorphisms of K that H induces on K. Suitable restrictions on the pair $(K, \bar{H})$ imply $H \in \mathfrak{X}$. We begin with two extremes: no restrictions on $\bar{H}$ in the first case and no restrictions on K in the second.

THEOREM 1. *Let K be a standard wreath product of two p-groups. Then $H \in \mathfrak{X}$.*

THEOREM 2. *Let $\bar{H}$ be a Sylow p-subgroup of* $\mathrm{Aut}(K)$. *Then $H \in \mathfrak{X}$.*

Our other results will impose restrictions on K as well as on $\bar{H}$. Since by a well-known theorem of P. Hall $K_i(H)$ is abelian, if the class of H is less than $2i$, H has in general several characteristic abelian subgroups. So we assume in the following that K is abelian. Then $\bar{H}$ can be considerably smaller than in Theorem 2.

THEOREM 3. *Let K be non-homogeneous abelian. If $\bar{H}$ contains $O_p(\mathrm{Aut}(K))$, the largest normal p-subgroup of* $\mathrm{Aut}(K)$, *then $H \in \mathfrak{X}$.*

Note that by Shoda [8] $|O_p(\mathrm{Aut}(K))| = p^m$, where

$$m = \sum_{i=1}^{r-1} (2n_i + n_{i+1} + \cdots + n_r)(n_{i+1} + \cdots + n_r)$$

and the homogeneous components of exponent p^i of K have order p^{in_i}. Since this is still a very large power of p, we analyze the structure of $\mathrm{Aut}(K)$ more precisely. Thus we obtain an independent proof of Shoda's results and additional information on the subgroup structure of $\mathrm{Aut}(K)$. This gives the following improvement of Theorem 3.

THEOREM 4. *Let K be non-homogeneous abelian with homogeneous components of exponent p^i and order p^{in_i}, $n_i \geqq 0$ for $i = 1, \cdots, r$. If $n_i \neq 0$ and $\bar{H}$ contains one of the normal p-subgroups A_i and B_i of* $\mathrm{Aut}(G)$, *then $H \in \mathfrak{X}$. The groups A_i and B_i both have order p^{m_i}, where*

$$m_i = (n_i + \cdots + n_r)(n_{i+1} + \cdots + n_r) + \sum_{j=1}^{i-1} (2n_j + n_{j+1} + \cdots + n_r)(n_{j+1} + \cdots + n_r).$$

If we set

$$C_i = C_{\mathrm{Aut}(K)}(\mho_i(K)) \cap \bigcap_{j=1}^{i} C_{\mathrm{Aut}(K)}(\Omega_1(K) \cap \mho_{j-1}(K)/\Omega_1(K) \cap \mho_j(K)),$$

the definitions of A_i *and* B_i *are*

$$A_i = C_i \cap C_{\mathrm{Aut}(G)}(\mho_{i-1}(K)/\mho_i(K)) \text{ and}$$

$$B_i = C_i \cap C_{\mathrm{Aut}(G)}(\Omega_1(K) \cap \mho_{i-1}(K)).$$

Obviously the best bound in Theorem 4 is obtained if $n_1 \neq 0$. We formulate this as a corollary.

COROLLARY 1. *Let* K *be as in Theorem 4 and* $n_1 \neq 0$. *Then* $H \in \mathfrak{X}$ *if* $|\bar{H}| \geqq p^{m_1}$, $m_1 = (\Sigma_{i=2}^{r} n_i)(\Sigma_{i=1}^{r} n_i)$, *and either* $\Omega_1(K)\mho_1(K) \leqq Z(H)$ *or* $[K,H] \leqq \Omega_1(K) \cap \mho_1(K)$.

We point out that in Corollary 1 no explicit knowledge of $\mathrm{Aut}(K)$ is required. It can be checked in the group itself whether the hypotheses are satisfied or not. This is also a feature of our further results. In addition they also apply to the case of homogeneous abelian K which has been excluded so far.

THEOREM 5. *Let* $[K,H] \leqq C_K(H) = K \cap Z(H)$. *Then* $H \in \mathfrak{X}$, *if*

(i) *there is a complement* T *of* $C_K(H)$ *in* K *such that* $[K,H]T$ *is characteristic in* H *and the number of complements of* $[K,H]$ *in* $[K,H]T$ *is at most* $|\bar{H}|$, *or*

(ii) *there is a complement* T *of* $[K,H]$ *in* K *such that* $C_K(H) \cap T$ *is characteristic in* H *and the number complements of* $C_K(H)/C_K(H) \cap T$ *in* $K/C_K(H) \cap T$ *is at most* $|\bar{H}|$.

Of course the condition that $[K,H]T$ or $C_K(H) \cap T$ be characteristic in H is a heavy restriction. If $[K,H] = C_K(H)$ this hypothesis is automatically satisfied. So we have an important special case.

COROLLARY 2. *Let* $[K,H] = C_K(H)$ *and* $[K,H]$ *be complemented in* K. *Then* $H \in \mathfrak{X}$, *if the number of complements is at most* $|\bar{H}|$. *Especially if* $[K,H] = C_K(H)$ *is complemented in* K *and* $|\bar{H}| \geqq |[K,H]|^d$, *where* $|K/[K,H]| = p^d$, *then* $H \in \mathfrak{X}$.

From these results one can deduce the above cited Theorem of Stitzinger, Hill and Wright as well as the following dualization.

THEOREM 6. *If* $|K/[K,H]| = p$ *and* $|K| \neq p$, *then* $H \in \mathfrak{X}$.

Finally we give some examples of p-groups that lie in $\mathfrak{X}$ and have a large center. Of course one can apply Theorem 1 to construct such examples, but these have very large orders. Instead we construct p-groups which satisfy the hypotheses of Theorem 6 and Corollary 2 and have minimal order with respect to these properties. In addition the center may be any elementary abelian p-group and every automorphism of the center extends to an automorphism of the whole group. In particular no proper subgroup of the center is characteristic in the whole group.

3. Notation

Our notation is mostly standard, we refer to Huppert's book [4]. For any subgroup U of a group G we define the normalizer $N_{\mathrm{Aut}(G)}(U)$ of U in the automorphism group $\mathrm{Aut}(G)$ to be the subgroup of those automorphisms that leave U invariant. If F is a factor of G, the centralizer $C_{\mathrm{Aut}(G)}(F)$ of F in $\mathrm{Aut}(G)$ is the subgroup of those automorphisms that induce the identity map on F.

4. Stabilizing automorphism groups

Most of our results are based on the observation that some automorphism groups of split extensions also split.

LEMMA 1. *Let M and N be normal subgroups of a group G such that $M \leqq Z(N)$.*

(i) *If N has a complement H in G define $\mathfrak{H}$ to be the set of all complements of M in MH.*

(ii) *If M has a complement H in G define $\mathfrak{H}$ to be the set of those complements U of M, for which $N \cap H = N \cap U$.*

Then in each case $D = C_{\mathrm{Aut}(G)}(N) \cap C_{\mathrm{Aut}(G)}(G/M)$ acts faithfully as a regular group of permutations on $\mathfrak{H}$. In particular $|D| = |\mathfrak{H}|$ and any automorphism group A of G that contains D and acts on $\mathfrak{H}$ splits over D.

PROOF. First we show that in eaoh case D acts on $\mathfrak{H}$, which is not empty, since $H \in \mathfrak{H}$. Now let $H_1 \in \mathfrak{H}$. In case (i) for each $\alpha \in D$ we have $H_1^\alpha \cap M = (H_1 \cap M)^\alpha = 1$ and $H_1^\alpha M = (H_1 M)^\alpha = (HM)^\alpha = HM$, since α centralizes G/M. So $H_1^\alpha \in \mathfrak{H}$. In case (ii) $MH_1^\alpha = (MH_1)^\alpha = G$ and $N \cap H_1^\alpha = (N \cap H_1)^\alpha = N \cap H_1$, since α centralizes M. So again $H_1^\alpha \in \mathfrak{H}$.

Next we observe that D acts fixpointfreely on $\mathfrak{H}$. In case (i) H_1 is again a complement of N in G: We have $NH_1 = NMH_1 = NMH = G$, and since $(N \cap H_1)M = N \cap H_1 M = N \cap HM = (N \cap H)M = M$, also $N \cap H_1 \leqq M \cap H_1 = 1$. Therefore any $\alpha \in N_D(H_1)$ centralizes H_1 and N and is thus trivial on G. In case (ii) H_1 is by definition a complement of M in G. Therefore again any $\alpha \in N_D(H_1)$ centralizes H_1 and M and is thus trivial on G.

It remains to show that in each case D is transitive on $\mathfrak{H}$. So let H_1 and $H_2 \in \mathfrak{H}$. In case (i) for each $\bar{g} \in G/N$ and for $i = 1,2$ there is exactly one $h_i(\bar{g}) \in H_i$ such that $\bar{g} = Nh_i(\bar{g})$. Since even $Mh_1(\bar{g}) = Mh_2(\bar{g})$, we have $h_1(\bar{g})^{-1}h_2(\bar{g}) \in M \leqq Z(N)$. In case (ii) for each $\bar{g} \in G/M$ and for $i = 1,2$ there is exactly one $h_i(\bar{g}) \in H_i$ such that $\bar{g} = Mh_i(\bar{g})$. Since $Nh_1(\bar{g}) = Nh_2(\bar{g})$, we have $h_1(g)^{-1}h_2(g) \in N \leqq C_G(M)$ in this case. Now for each $g \in \bar{g}$ there is some $n \in N$, respectively $n \in M$ in case (ii), such that $g = nh_1(\bar{g})$. We define $\alpha : G \to G$ by $\alpha : nh_1(\bar{g}) \mapsto nh_2(\bar{g})$, where $Nh_1(\bar{g}) = Nh_2(\bar{g})$, respectively $Mh_1(\bar{g}) = Mh_2(\bar{g})$. Then α is bijective and we show that α is a homomorphism. If $g_1 = n_1h_1(\bar{g}_1)$ and $g_2 = n_2h_1(\bar{g}_2)$, then

$$\begin{aligned}(g_1g_2)^\alpha &= (n_1h_1(\bar{g}_1)\,n_2h_1(\bar{g}_2))^\alpha = (n_1n_2^{h_1(\bar{g}_1)^{-1}}h_1(\bar{g}_1)\,h_1(\bar{g}_2))^\alpha \\ &= (n_1n_2^{h_1(\bar{g}_1)^{-1}}h_1(\bar{g}_1\bar{g}_2))^\alpha = n_1n_2^{h_1(\bar{g}_1)^{-1}}h_2(\bar{g}_1\bar{g}_2) \\ &= n_1n_2^{h_1(\bar{g}_1)^{-1}}h_2(\bar{g}_1)\,h_2(\bar{g}_2) = n_1h_2(\bar{g}_1)\,n_2^{h_1(\bar{g}_1)^{-1}h_2(\bar{g}_1)}h_2(g_2) \\ &= n_1h_2(\bar{g}_1)\,n_2h_2(\bar{g}_2) = g_1^\alpha g_2^\alpha,\end{aligned}$$

since in each case n_2 commutes with $h_1(\bar{g}_1)^{-1}h_2(\bar{g}_1)$. So α is an automorphism of G which maps H_1 onto H_2 and induces the identity on N and G/M in both cases.

We remark that this result is a generalization of Hilfssatz VI 7.14 of Huppert [4].

For our applications of Lemma 1 we collect some additional information on the stabilizing group D.

Lemma 2. *Let $M \leqq N$ be two normal subgroups of a group G and let $U \leqq D = C_{\mathrm{Aut}(G)}(N) \cap C_{\mathrm{Aut}(G)}(G/M)$.*

(i) *If G/N is generated by d elements, then D can be embedded into $Z(M) \times \cdots \times Z(M)$, where the direct product has d factors.*

(ii) *If $M \leqq Z(G)$ then $D \cong \mathrm{Hom}(G/N, M)$. If, in addition, $M = [G, U]$ and $N = C_K(U)$, $|M| = p^s$ and $|G/N| = p^r$, then the minimal number of generators of U is at least $m = \max(r/s, s/r)$.*

Proof. (i) follows from [5, prop. 1.1].

(ii) Each $\alpha \in D$ defines a homomorphism $f_\alpha : g \mapsto g^{-1}g^\alpha$ from G into M with $\ker f_\alpha \geqq N$. Obviously the map $\varphi : \alpha \mapsto f_\alpha$ is injective. On the other hand each homomorphism $f : G \to M$ with $\ker f \geqq N$ defines an automorphism $\alpha_f \in D$ by $\alpha_f : g \mapsto gf(g)$ for each $g \in G$. We claim that φ is a homomorphism. For all $\alpha, \beta \in D$ and $g \in G$ we have $g^{\alpha\beta} = (gf_\alpha(g))^\beta = gf_\beta(g)f_\alpha(g)$, since β centralizes M and $g^{\alpha\beta} = gf_{\alpha\beta}(g)$. Thus for all $g \in G$, $f_{\alpha\beta}(g) = f_\alpha(g)f_\beta(g)$. Therefore φ is an isomorphism from D onto the subgroup of those elements of $\mathrm{Hom}(G,M)$ that contain N in their kernel.

Let $U = \langle \alpha_1, \cdots, \alpha_n \rangle$ and $\varphi(\alpha_i) = f_i$, where φ is defined as in the proof of (ii). Since $N \leqq \ker f_i$, we have $|\mathrm{im} f_i| \leqq |G/N| = p^r$ for $i = 1, \cdots, n$. If we set $E = \langle \mathrm{im} f_i \mid i = 1, \cdots, n \rangle$, then $|E| \leqq p^{rn}$. Now U centralizes G/E such that $[G, U] \leqq E$. Therefore $p^s \leqq p^{rn}$, equivalently $s/r \leqq n$. on the other hand $\mathrm{im} f_i \leqq M$ implies $|G/\ker f_i| \leqq |M| = p^s$ for $i = 1, \cdots, n$. If we set $R = \bigcap_{i=1}^n \ker f_i$, then $|G/R| \leqq p^{ns}$. Now U centralizes R such that $R \leqq N$. Therefore $|G/R| \geqq p^r$ and $p^{ns} \geqq p^r$, equivalently $n \geqq r/s$. Thus the minimal number of generators of U is at least $m = \max(r/s, s/r)$.

5. Automorphism groups of abelian groups

Speiser [9] has analyzed the structure of the automorphism group of a finite abelian group by induction along the descending chain of the $\mho_i(G)$. This group-theoretic procedure allows to determine the action of certain subgroups of $\mathrm{Aut}(G)$ somewhat better than the ring-theoretic approach of Shoda [8]. We therefore follow Speiser's concept. We look for normal p-subgroups of $\mathrm{Aut}(G)$ that induce a transitive permutation group on a certain characteristic family of subgroups of G.

PROPOSITION 1. *Let $G = G_1 \times G_2$ be an abelian p-group with a homogeneous component G_1 of order p^{n_1} and exponent p. Let m be the minimal number of generators of G_2. Set $A = C_{\mathrm{Aut}(G)}(\mho_1(G))$. Then the two normal p-subgroups $A_1 = C_A(\Omega_1(G))$ and $A_2 = C_A(G/\mho_1(G))$ of $\mathrm{Aut}(G)$ are elementary abelian of order $p^{(n_1+m)m}$. The intersection $D = A_1 \cap A_2$ has order p^{m^2} and lies in $Z(A)$. A_1/D can be regarded as a regular permutation group on the set of all complements of $\Omega_1(G)\mho_1(G)/\mho_1(G)$ in $G/\mho_1(G)$. A_2/D can be regarded as a regular permutation group on the set of all complements of $\Omega_1(G) \cap \mho_1(G)$ in $\Omega_1(G)$. The product A_1A_2 is just $B = C_A(\Omega_1(G)/\Omega_1(G) \cap \mho_1(G))$, and A/B is isomorphic to $\mathrm{GL}(N_1, p)$.*

PROOF. We denote by $\mathfrak{H}_1$ the set of all complements of $\mho_1(G) \cap \Omega_1(G)$ in $\Omega_1(G)$ and by $\mathfrak{H}_2$ the set of all complements of $\Omega_1(G)\mho_1(G)/\mho_1(G)$ in $G/\mho_1(G)$. We claim that $G_1 \in \mathfrak{H}_1$ and $G_2/\mho_1(G) \in \mathfrak{H}_2$. First $\mho_1(G) = \mho_1(G_2) \leqq G_2$ and $\Omega_1(G_2) \leqq \mho_1(G_2)$, since G_2 has no homogeneous component of exponent p. Therefore $(\Omega_1(G) \cap \mho_1(G))G_1 = \mho_1(G)G_1 \cap \Omega_1(G) = \Omega_1(\mho_1(G))\Omega_1(G_1) = \Omega_1(G_2)G_1 = \Omega_1(G)$ and $\Omega_1(G) \cap \mho_1(G) \cap G_1 = \mho_1(G_1) = 1$, such that $G_1 \in \mathfrak{H}_1$.

Similarly $G_2 \cap \Omega_1(G)\mho_1(G) = (\Omega_1(G) \cap G_2)\mho_1(G) = \Omega_1(G_2)\mho_1(G) = \mho_1(G)$ and $G_2\Omega_1(G)\mho_1(G) = G_2G_1 = G$, such that $G_2/\mho_1(G) \in \mathfrak{H}_2$.

Now $\Omega_1(G)/\Omega_1(G) \cap \mho_1(G)$ is isomorphic to G_1 such that A induces on the factor an automorphism group which is contained in $\mathrm{Gl}(n_1, p)$. Each automorphism of G_1 can be extended to an automorphism of G that centralizes G_2 and thus lies in A. Therefore $A/B \cong \mathrm{Gl}(n_1, p)$. Now B centralizes $\Omega_1(G) \cap \mho_1(G)$ and $\Omega_1(G)/\Omega_1(G) \cap \mho_1(G)$. Therefore by Lemma 1, B acts fixpointfreely on $\mathfrak{H}_1$. Since B contains $C_1 = C_{\mathrm{Aut}(G)}(G_2) \cap C_{\mathrm{Aut}(G)}(G/\Omega_1(G) \cap \mho_1(G))$, which is already regular on $\mathfrak{H}_1$ by Lemma 1, also B acts regularly on $\mathfrak{H}_1$. As well B centralizes $G/\Omega_1(G)\mho_1(G)$ and $\mho_1(G)\Omega_1(G)/\mho_1(G)$, since the latter factor is covered by $\Omega_1(G)/\Omega_1(G) \cap \mho_1(G)$. Therefore again by Lemma 1, B acts fixpointfreely on $\mathfrak{H}_2$. Since B contains $C_2 = C_{\mathrm{Aut}(G)}(G/G_1) \cap C_{\mathrm{Aut}(G)}(\mho_1(G)\Omega_1(G))$, which is already regular on $\mathfrak{H}_2$ by Lemma 1, also B acts regularly on $\mathfrak{H}_2$. By Lemma 2 (ii) we obtain $|B/A_1| = p^{n_1m} = |B/A_2|$. Now C_1 is already contained in A_2 and C_2 is contained in A_1. Therefore A_2 covers B/A_1 and A_1 covers B/A_2 such that $A_1A_2 = B$ and $|B/A_1 \cap A_2| = p^{2n_1m}$. In addition A_1 is regular on $\mathfrak{H}_2$ and A_2 is regular on $\mathfrak{H}_1$. Finally we have $C_{\mathrm{Aut}(G)}(\mho_1(G)) = C_{\mathrm{Aut}(G)}(G/\Omega_1(G))$, see Speiser [9, p. 129], such that $A_1 = C_{\mathrm{Aut}(G)}(\Omega_1(G)) \cap C_{\mathrm{Aut}(G)}(G/\Omega_1(G))$ and $A_2 = C_{\mathrm{Aut}(G)}(\mho_1(G)) \cap C_{\mathrm{Aut}(G)}(G/\mho_1(G))$ are elementary abelian by Lemma 2. In addition, the Three-Subgroup-Lemma yields

$$[D, A, G] \leqq [A, G, D][G, D, A] \leqq [\Omega_1(G), D][\Omega_1(G) \cap \mho_1(G), A] = 1$$

such that $D \leqq Z(A)$.

Observing that every automorphism of $\mho_1(G)$ can be extended to an automorphism of G, one has $\mathrm{Aut}(G)/C_{\mathrm{Aut}(G)}(\mho_1(G)) \cong \mathrm{Aut}(\mho_1(G))$. So inductive application of Proposition 1 gives a description of $\mathrm{Aut}(G)$.

Proposition 2. *Let $G = G_1 \times \cdots \times G_r$ be an abelian p-group with homogeneous components G_i of exponent p^i and order p^{in_i} for $i = 1, \cdots, r$. We set $F_i = \Omega_1(G) \cap \mho_{i-1}(G)/\Omega_1(G) \cap \mho_i(G)$ and $\mathfrak{H}_i = \{H_i \mid H_i$ is a complement of $\Omega_1(G) \cap \mho_i(G)$ in $\Omega_1(G) \cap \mho_{i-1}(G)\}$, $\mathfrak{K}_i = \{K_i \mid K_i$ is a complement of $\mho_i(G)(\Omega_1(G) \cap \mho_{i-1}(G))/\mho_i(G)$ in $\mho_{i-1}(G)/\mho_i(G)\}$ for $i = 1, \cdots, r$. Then* $\mathrm{Aut}(G)$ *contains normal p-subgroups A_i and B_i that act transitively on $\mathfrak{H}_i$ respectively $\mathfrak{K}_i$ for $i = 1, \cdots, r$. If we set $C_i = C_{\mathrm{Aut}(G)}(\mho_i(G)) \cap \bigcap_{j=1}^{i} C_{\mathrm{Aut}(G)}(F_j)$,* then *$A_i = C_i \cap C_{\mathrm{Aut}(G)}(\mho_{i-1}(G)/\mho_i(G))$* and *$B_i = C_i \cap C_{\mathrm{Aut}(G)}(\Omega_1(G) \cap \mho_{i-1}(G))$*. The orders are

$$|A_i| = |B_i| = p^{m_i}, m_i = \sum_{j=1}^{i-1} (2n_j + n_{j+1} + \cdots + n_r)(n_{j+1} + \cdots + n_r)$$
$$+ (n_i + n_{i+1} + \cdots + n_r)(n_{i+1} + \cdots + n_r).$$

PROOF. We proceed by induction on i. For $i = 1$ we apply Proposition 1. So let Proposition 2 be proved for $\mho_1(G)$. Then $\mathrm{Aut}(G)/C_{\mathrm{Aut}(G)}(\mho_1(G))$ contains normal p-subgroups $A_i^*/C_{\mathrm{Aut}(G)}(\mho_1(G))$ and $B_i^*/C_{\mathrm{Aut}(G)}(\mho_1(G))$ as claimed. The definitions of A_i^* and B_i^* are

$$A_i^* = C_{\mathrm{Aut}(G)}(\mho_i(\mho_1(G))) \cap C_{\mathrm{Aut}(G)}(\mho_{i-1}(\mho_1(G))/\mho_i(\mho_1(G))) \cap \bigcap_{j=2}^{i} C_{\mathrm{Aut}(G)}(F_j)$$

$$= C_{\mathrm{Aut}(G)}(\mho_{i+1}(G)) \cap C_{\mathrm{Aut}(G)}(\mho_i(G)/\mho_{i+1}(G)) \cap \bigcap_{j=2}^{i} C_{\mathrm{Aut}(G)}(F_j)$$

and similarly

$$B_i^* = C_{\mathrm{Aut}(G)}(\mho_{i+1}(G)) \cap C_{\mathrm{Aut}(G)}(\Omega_1(G) \cap \mho_i(G)) \cap \bigcap_{j=2}^{i} C_{\mathrm{Aut}(G)}(F_j).$$

Now $A_{i+1} = A_i^* \cap C_{\mathrm{Aut}(G)}(F_1)$ and $B_{i+1} = B_i^* \cap C_{\mathrm{Aut}(G)}(F_1)$ for $i = 1, \cdots, r-1$, and A_1 and B_1 are just A_1 and A_2 in Proposition 1. Since $C_{\mathrm{Aut}(G)}(\mho_1(G))$ is contained in A_i^* and B_i^*, both groups induce on F_1 the full automorphism group $\mathrm{GL}(n_1, p)$ of F_1. Therefore $A_i^* = A_{i+1} C_{\mathrm{Aut}(G)}(\mho_1(G))$ and $B_i^* = B_{i+1} C_{\mathrm{Aut}(G)}(\mho_1(G))$ and $A_{i+1} \cap C_{\mathrm{Aut}(G)}(\mho_1(G)) = C_1 = B_{i+1} \cap C_{\mathrm{Aut}(G)}(\mho_1(G))$. By inductive hypothesis A_i^* is transitive on $\mathfrak{H}_{i+1}$, and since $C_{\mathrm{Aut}(G)}(\mho_1(G))$ acts trivial on $\mathfrak{H}_{i+1}$, also A_{i+1} is transitive on $\mathfrak{H}_{i+1}$. By the same argument B_{i+1} is transitive on $\mathfrak{K}_{i+1}$. The formula for the orders of the A_i and B_i easily follows by induction from Proposition 1.

REMARK. For $i = r$ in Proposition 2 we obtain $C_r = A_r = B_r = O_p(\mathrm{Aut}(G))$ and $|\mathrm{Aut}(G)/C_r| = \Pi_{i=1}^r |\mathrm{GL}(n_i, p)|$. Since $\mathrm{Aut}(G)$ has a subgroup isomorphic to $\Pi_{i=1}^r \mathrm{Aut}(G_i)$, which covers $\mathrm{Aut}(G)/C_r$, in fact $\mathrm{Aut}(G)/C_r \cong \Pi_{i=1}^r \mathrm{GL}(n_i, p)$.

6. Proofs of the Theorems

Most of our results are consequences of the following well-known fact.

LEMMA 3. *Let G act on a set $\mathfrak{H}$ and let N be a normal subgroup of G that acts transitively on $\mathfrak{H}$. Then for each $H \in \mathfrak{H}$ we have $G = NN_G(H)$. In particular $|\mathfrak{H}| > 1$ implies $N \not\leq \Phi(G)$.*

This follows easily by the Frattini argument.

Now for all of our proofs we assume that our p-group H is a normal subgroup of a group G contained in $\Phi(G)$. We denote by f the natural homomorphism from G into $\mathrm{Aut}(K)$ such that $f(H)=\bar{H}\leqq f(\Phi(G))\leqq\Phi(f(G))$ by Gaschütz's Satz 3 [1]. So we only have to find a contradiction in this situation for each theorem.

Proof of Theorem 1. Let $K=AwrB$, where A and B are p-groups. If B and A have order two, then K is the dihedral group of order 8 and $K\in\mathfrak{X}$, in particular $H\in\mathfrak{X}$. In all other cases by P. M. Neumann's theorem 9.1 [7] the base group of K is characteristic in K. Therefore $\mathrm{Aut}(G)$ acts as a permutation group on the set $\mathfrak{H}$ of all complements of the base group. By P. M. Neumann's theorem 10.1 [7] $\mathrm{In}(K)$ is transitive on $\mathfrak{H}$. Therefore Lemma 3 gives the contradiction.

Proof of Theorem 2. By Huppert [4, III 3.8] each prime divisor of $|\Phi(f(G))|$ divides $|f(G)/\Phi(f(G))|$. But the hypothesis implies that $\Phi(f(G))$ contains a Sylow p-subgroup of $f(G)$, a contradiction.

Proof of Theorem 3. By the remark after the proof of Proposition 2, $O_p(\mathrm{Aut}(K))$ has a supplement in $\mathrm{Aut}(K)$ if K is not homogeneous. So by our hypothesis H has a supplement in $f(G)$, a contradiction.

Proof of Theorem 4. If $n_i\neq 0$, then $\mathfrak{H}_i$ and $\mathfrak{K}_i$ have more than one element. Since $\mathrm{Aut}(K)$ acts on $\mathfrak{H}_i$ and $\mathfrak{K}_i$, also $f(G)$ acts on these sets. If $A_i\leqq\bar{H}$ or $B_i\leqq\bar{H}$, then by Proposition 2 $\bar{H}$ is transitive on one of these sets. Therefore Lemma 3 gives the contradiction.

Proof of Corollary 1. The hypotheses imply either $\bar{H}=A_1$ or $\bar{H}=B_1$. Therefore we have a special case of Theorem 4.

Proof of Theorem 5. Obviously

$$\bar{H}\leqq D=C_{\mathrm{Aut}(K)}(C_K(H))\cap C_{\mathrm{Aut}(K)}(K/[K,H]).$$

Our hypotheses (i) and (ii) correspond to (i) and (ii) in Lemma 1 such that D acts regularly on the set of complements of $[K,H]$ in $[K,H]T$ in case (i), respectively of $C_K(H)/C_K(H)\cap T$ in $K/C_K(H)\cap T$ in case (ii). Thus our hypothesis on the order of $f(H)$ gives $f(H)=D$ in each case. The hypothesis that $[K,H]T$, respectively $C_K(H)\cap T$, be characteristic in G yields that G acts on $\mathfrak{H}$. Therefore Lemma 3 gives the contradiction.

We have to prove the second part of Corollary 2. So let $|\bar{H}|\geqq|[K,H]|^d$, where $|K/[K,H]|=p^d$. Since H centralizes $K/[K,H]$ and $[K,H]$, $\bar{H}$ lies in $D=C_{\mathrm{Aut}(K)}([K,H])\cap C_{\mathrm{Aut}(K)}(K/[K,H])$. By Lemma 2 (i) $|D|\leqq|[K,H]|^n$, where n is the minimal number of generators of $K/[K,H]$. Since $|[K,H]|^d \leqq |\bar{H}| \leqq |D| \leqq |[K,H]|^n$, we have $n = d$ and $\bar{H} = D$. By Lemma 1, D is equal to the number of complements of $[K,H]$ in K. So we can apply Theorem 5.

PROOF OF THEOREM 6. If $C \leqq K$ and C is normal in G, then $\Phi(G/C) = \Phi(G)/C$, see Gaschütz [1, Satz 2]. Therefore also $H/C \leqq \Phi(G/C)$ and we can assume $C = 1$. If $[K,H] = \Phi(K)$, then K is cyclic and has only characteristic subgroups. Then we set $\mho_2(K) = 1$. We have $|K| = p^2$ and $\bar{H}$ is a Sylow p-subgroup of $\mathrm{Aut}(K)$. Therefore we can apply Theorem 2. If $[K,H] \leqq \Phi(K)$, then we set $\Phi(K) = 1$ and K is elementary abelian. Now $[K,H,H] \leqq [K,H]$ and $[K,H,H]$ is normal in G, since K and H are normal in G. Therefore we assume $[K,H,H] = 1$ and we have $[K,H] = C_K(H)$ and $|K/[K,H]| = p$. Now Lemma 2 (iii) shows that $|\bar{H}|$ is at least $|[K,H]|$. Since K is elementary abelian, we can apply Corollary 2 to obtain the result.

In a dual way we obtain a proof for the Theorem of Stitzinger, Hill and Wright, cited in the introduction. Instead of $\Phi(A)$ we consider $\Omega_1(A)$. If $A \cap Z(H) = \Omega_1(A)$, then A is cyclic and we can assume $|A| = p^2$. Since $A \not\leqq Z(H)$, H induces on A an automorphism group of order p. Then Theorem 2 yields a contradiction. If $\Omega_1(A) > A \cap Z(H)$, then we set $K = \Omega_1(A)$. Now $[K,H,H] \leqq [Z_2(H),H,H] = 1$, and $1 \neq [K,H] \leqq C_K(H) = K \cap Z(H) = A \cap Z(H)$ implies $[K,H] = C_K(H)$ and $|[K,H]| = p$. Again Lemma 2 shows that $|\bar{H}| = |K/[K,H]|$, and we can apply Corollary 2 to obtain the result.

7. Examples of special p-groups that lie in $\mathfrak{X}$

Our first example settles the case $p = 2$. Here we can construct a semidirect product which has the required properties.

Let V be elementary abelian of order 2^n, $n \geqq 3$, and W a subgroup of order 2^{n-1}. Then by Lemma 2 (ii) the automorphism group $D = C_{\mathrm{Aut}(V)}(W)$ is isomorphic to W. Let G be the semidirect product VD, where D acts on V in the natural way. We claim that V is a characteristic subgroup of G So let $\alpha \in \mathrm{Aut}(G)$ and $K = V^\alpha$. Then $K = C_G(K)$, since $V = C_G(V)$. Now $WD = W \times D$ is abelian of order $|W|^2 \neq |V|$. So K cannot be contained in WD. Let $g \in K$ and $g \notin WD$. Then $g = vd$ with some $v \in V \backslash W$ and some $d \in D$. Since $W = C_V(D)$, we have $[v,d] \neq 1$ for $d \neq 1$. On the other hand $[v,d] = (vd)^2 = 1$, since K is elementary abelian. So $d = 1$ and $V = K$. As we have shown, V is a characteristic elementary abelian subgroup of G with $|V/[V,G]| = 2$. So Theorem 6 applies.

Obviously $W = Z(G)$ can be chosen arbitrary large. We show that each automorphism α of W can be extended to an automorphism of G. First α extends to an automorphism β of V, which leaves W invariant. Because D is a normal subgroup of $N_{\mathrm{Aut}(V)}(W)$, β induces by conjugation an automorphism γ on D. Now the pair (β, γ) defines an automorphism δ of G by $(vd)^\delta = v^\beta d^\gamma$ for each $vd \in VD = G$, as one can compute directly or deduce from Kung Wei Yang [11, theor. 2]. Thus no subgroup of W is characteristic in G.

We remark that the case $n = 2$, excluded here, gives the dihedral group of order 8 which is also in $\mathfrak{X}$.

For the case $p \neq 2$ we have another type of special p-groups with the required properties.

Let G be a class two group generated by $x_1, \cdots, x_n$ subject to the relations $x_i^p = [x_1, x_i]$ for $i = 1, \cdots, n$ and $[x_i, x_j] = 1$ for $i \neq 1 \neq j$. Then G/G' is elementary abelian of order p^n and G' is elementary abelian of order p^{n-1}. Since $p \neq 2$ and $\mathrm{cl}(G) = 2$, the map $\theta : g \mapsto g^p$ for each $g \in G$ is a homomorphism from G onto G' whose kernel V is a characteristic subgroup of G. Obviously $V = \langle G', x_1 \rangle$ and V is elementary abelian. By the relations we have $|V/[V, G]| = p$ and $[V, G] = C_V(G)$ such that we can apply Theorem 6 and Corollary 2 to see that $G \in \mathfrak{X}$. Since $[V, G] \leqq Z(G) \leqq C_G(V) = V$ and $Z(G) \neq V$, we have $Z(G) = [V, G]$ elementary abelian of order p^{n-1}. Obviously $Z(G)$ can be chosen arbitrary large, and as we show no proper subgroup of $Z(G)$ is characteristic in G.

On $H = \langle x_2, \cdots, x_n \rangle$ the map $\theta : h \mapsto h^p$ coincides with the map $k \mapsto [x_1, h]$, so that $h^p = [x_1, h]$ for all $h \in H$. If $\{y_2, \cdots, y_n\}$ is another basis for H, then also $x_1, y_2, \cdots, y_n$ are generators for G and satisfy the same relations as $x_1, \cdots, x_n$. So there exists an automorphism of G sending $x_i \mapsto y_i$ for $i = 2, \cdots, n$. Therefore $\mathrm{Aut}(G)/C_{\mathrm{Aut}(G)}(G/V)$ is isomorphic to $\mathrm{GL}(n-1, p)$. Now the homomorphism θ permutes with all automorphisms of G. Thus, $C_{\mathrm{Aut}(G)}(G/V) = C_{\mathrm{Aut}(G)}(Z(G))$ and $\mathrm{Aut}(G)/C_{\mathrm{Aut}(G)}(Z(G)) \cong \mathrm{GL}(n-1, p) \cong \mathrm{Aut}(Z(G))$.

The author is indebted to the referee for a shorter, more elegant proof of the fact that every automorphism of $Z(G)$ extends to an automorphism of the whole group in our last example.

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